the review of income and wealth

# Appendix: Parameter combinations determining the Density Modality of the Elliptical LC (Section 4.3)

In order to analyze the sign of the third derivative in (20)

$$L'''(\pi, a, b, d) = \frac{\frac{3}{16}(4\alpha e^2 - \beta^2)(2\alpha\pi + \beta)}{\sqrt{(\alpha\pi^2 + \beta\pi + e^2)^5}}$$

and hence the density modality of the Elliptical LC, the four parameter conditions $\alpha = b^2 - 4a < 0$, $a + b + d + 1 = -e > 0$, $d \geq 0$ and $a + d - 1 \geq 0$ need to be kept in mind. The positivity of the denominator and the negativity of the factor $4\alpha e^2 - \beta^2$ imply that the sign of $L'''$ is the opposite sign of $2\alpha\pi + \beta = 2(b^2 - 4a)\pi + 2be - 4d$. Now one can distinguish several cases, which, taken together, give the result in the main text:

- For $b \geq 0$, $2(b^2 - 4a)\pi + 2be - 4d$ is always negative, implying a positive third derivative and thus a zeromodal density.
- For $b < 0$, the sign of $2(b^2 - 4a)\pi + 2be - 4d$ depends on the precise value of $b$. If it is sign-changing, it will reach the value of zero at the mode $\widetilde{\pi} = \frac{2d - be}{b^2 - 4a}$. $\widetilde{\pi}$ must fall within $(0, 1)$, otherwise there is no valid mode and the density is zeromodal just as for $b \geq 0$. The values of $b$ leading to $\widetilde{\pi} = 0$ and $\widetilde{\pi} = 1$ can be considered boundary values separating the unimodal from the zeromodal parameter range.
- The boundary $\widetilde{\pi} = 0$ can be established at

$$2d - be = 0 \tag{A.1}$$

$$b = -\frac{a + d + 1}{2} \pm \sqrt{\left(\frac{a + d + 1}{2}\right)^2 - 2d} \tag{A.2}$$

  If $b < 0$ is larger than $-\frac{a+d+1}{2} + \sqrt{\left(\frac{a+d+1}{2}\right)^2 - 2d}$, $\widetilde{\pi}$ is negative, thus there will be no valid sign-change and the density is zeromodal (Example: $a = 5, b = -0.5, d = 3$). $b < -\frac{a+d+1}{2} + \sqrt{\left(\frac{a+d+1}{2}\right)^2 - 2d}$ constitutes a condition for $b$ to be "sufficiently negative" for a sign-change.[1]

- The boundary $\widetilde{\pi} = 1$ can be established at

$$b = \frac{-4a - 2d}{a + d + 1}. \tag{A.3}$$

  If $b < 0$ is smaller than $\frac{-4a-2d}{a+d+1}$, $\widetilde{\pi}$ is larger than 1, thus there is no valid sign-change. In this case the third derivative is always negative (because $2\alpha\pi + \beta$ is positive),

[1] Only the larger of the two solutions of (2) is relevant because of the other stated restrictions on the parameter space.

implying an upward-sloping zeromodal density, to be neglected in practice (Example: $a = 2.5$, $b = -3$, $d = 0$). So $b > \frac{-4a-2d}{a+d+1}$ constitutes a condition for $b$ not to be "too negative" for a sign-change.