Socio-Economic Aspects of Urban Development

Unit 3: An Urban Economic Model of Rental Prices, Apartment Size, and Population Density

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An Urban Economic Model

- Cities are complex systems. A model can help us understand the underlying economic mechanisms and connections.
- Where do people live within cities?
- How are rental prices and population density related?
- Why are some cities more spatially extended than others?
- We will answer these questions based on
  - Standard model in urban economics based on William Alonso (1964), Edwin Mills (1967) und Richard Muth (1969). Very good technical summary by Brueckner (1987)
  - Further reading on these and other models written in an accessible way: Textbooks by O'Sullivan (2011) and Brueckner (2011)

Advantages and Disadvantages of Urban Economic Models

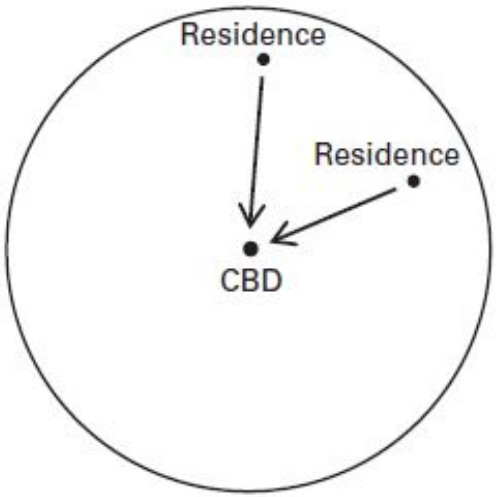
- Advantages of economic models for urban analysis
  - Focusing on the most relevant aspects
  - Representation of connections and relations
  - Uncovering chains of cause and effect
  - Mathematical formulas and graphical diagrams allow predictions and analysis
- ... and the corresponding disadvantages:
  - How realistic are the assumptions? Simplifications are necessary to reduce complexity. But do they limit the goodness of the predictions? How complex does the model need to be to include all the relationships important to the question at hand?
    - ★ "A map at a scale of 1:1 is meaningless."
    - ★ "Everything should be as simple as possible, but not simpler." (Albert Einstein)
- This unit focuses on the model - the next unit takes it to real-world data!

Main Assumptions of the Model (1/2)

- The standard Alonso-Mills-Muth model is based on very simple assumptions, but it can explain many empirical facts concerning population density, rental prices and commuting behavior
  - It remains relevant in urban economic research. Many extensions with specific scenarios and additional features!
- Main Assumptions:
  - All residents of the city are identical and have the same income.
  - They consume housing and one consumption good. The consumption good has a fixed price at all locations within the city, but rental prices vary.
  - The city is monocentric.
  - Land is owned by external landlords (or the state). Construction firms lease the land and build houses under perfect competition.
  - Residents rent/buy their apartments from the construction firms. No difference between renting or buying in this simple model.

Main Assumptions of the Standard Model (2/2)

- All jobs are located in the city center (Central Business District, CBD). This causes commuting costs, when people live at a distance x from the city center.



Demand Side: Urban Residents

- Residents have an income y which they spend on
  - the consumption good c with a numeraire price of 1
  - housing q, measured in square meters of floor space, at the price p per square meter. The rental price p is a function of distance x to the city center, measured in kilometers.
- Residents derive a utility of $v(c, q) = u$ from consuming the consumption good c and living space q.
- Residents have to pay a transport cost of $\tau \cdot x$ to commute to the city center. This means that people living on the outskirts have higher transport costs.

Demand Side: Urban Residents

- Equilibrium Condition

Theorem

Consumers must be equally well off at all locations, achieving the same utility regardless of where they live in the city.

- Why does utility have to be the same everywhere?
  - Arbitrage: Otherwise, residents of an area with low utility might benefit from moving to an area where they can achieve higher utility. There would be no equilibrium.
  - We know that residents on the outskirts of the city have higher commuting costs. That needs to be offset by a compensating factor that makes living there attractive: Lower rents p allow for larger apartments q. $\Rightarrow$ We will prove that p decreases with x and q increases with x.

Demand Side: Utility Maximization

- The residents have a **Budget Constraint**:

$$y - \tau \cdot x = p(x) \cdot q + c \cdot 1 \tag{1}$$

- Given their budget constraint, they **maximize their utility**:

$$\max_x = v(c, q) = v(y - \tau \cdot x - p(x) \cdot q, q) = u \tag{2}$$

- **First order condition** of the maximization problem: Marginal rate of substitution of housing q und consumption c must equal their price ratios of p and 1:

$$\frac{\frac{\partial v(c,q)}{\partial q}}{\frac{\partial v(c,q)}{\partial c}} = \frac{p}{1} \tag{3}$$

Demand Side: Utility Maximization

- Because we are interested in how the rental price p depends on distance x to the city center, we take the total differential of

$$v(c, q) = u \tag{4}$$

In the end, these variables are all functions of x.

$$\frac{\partial v(c,q)}{\partial c} \cdot \frac{\partial c}{\partial x} + \frac{\partial v(c,q)}{\partial q} \cdot \frac{\partial q}{\partial x} = 0 \tag{5}$$

- We have to take into account that consumption c depends on distance x to the city center via the budget constraint ($c = y - \tau \cdot x - p \cdot q$):

$$\frac{\partial v(c,q)}{\partial c} \cdot \left(-\tau - \frac{\partial p(x)}{\partial x} q(x) - p(x) \frac{\partial q(x)}{\partial x}\right) + \frac{\partial v(c,q)}{\partial q} \cdot \frac{\partial q(x)}{\partial x} = 0 \tag{6}$$

Demand Side: The Alonso-Muth Condition

- We divide (6) by $\frac{\partial v(c,q)}{\partial c}$ and recover the marginal rate of substitution from (3) which we can substitute with p:

$$-\tau - \frac{\partial p(x)}{\partial x} \cdot q(x) - p(x) \cdot \frac{\partial q(x)}{\partial x} + \frac{\frac{\partial v(c,q)}{\partial q(x)}}{\frac{\partial v(c,q)}{\partial c}} \cdot \frac{\partial q(x)}{\partial x} = 0 \quad (7)$$

$$-\tau - \frac{\partial p(x)}{\partial x} \cdot q(x) - p(x) \cdot \frac{\partial q(x)}{\partial x} + p(x) \cdot \frac{\partial q(x)}{\partial x} = 0 \quad (8)$$

- This leads us to

$$\tau + \frac{\partial p(x)}{\partial x} \cdot q(x) = 0 \quad (9)$$

$$\frac{\partial p(x)}{\partial x} = \frac{-\tau}{q(x)} < 0 \quad (10)$$

- (10) is the **Alonso-Muth Condition**.

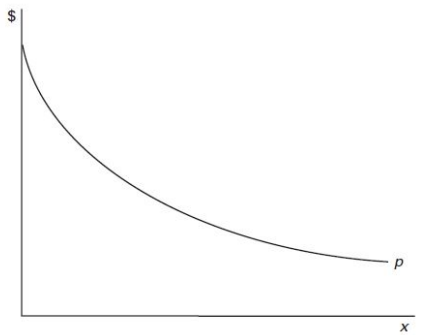
Rental Price as a Negative Function of Distance

- The Alonso-Muth Mills Condition

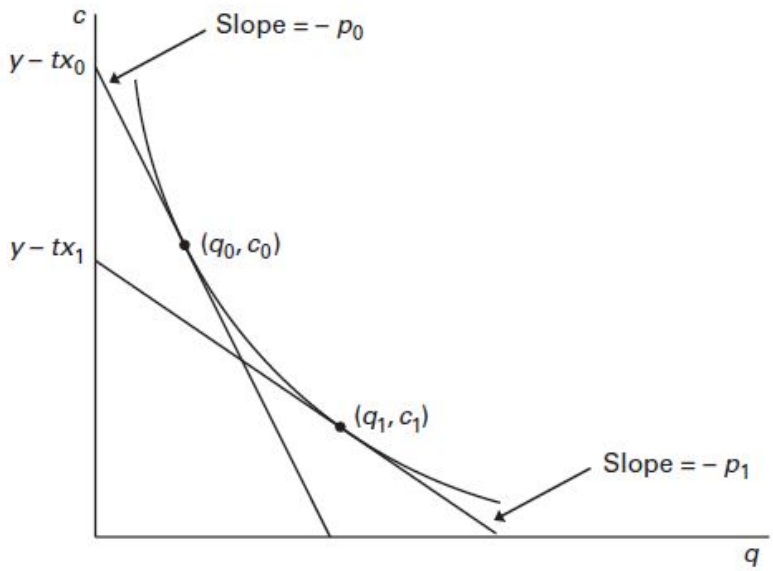
$$\frac{\partial p(x)}{\partial x} = \frac{-\tau}{q(x)} < 0 \tag{11}$$

says: **The rental price gradient, $\frac{\partial p(x)}{\partial x}$, the dependence of the rental rate on distance to the city center, is negative**. This is **Result #1** of the Alonso-Muth-Mills model!

- Convex function: The rental price declines first steeply and then less steeply with growing distance to the city center. In inner-city locations, a larger decline in the price per square meter is needed to compensate for one more km of commuting (than would be the case on the outskirts where commutes are long anyways).



Demand Side: Combinations of Housing and Consumption



- City center resident 0 has a steep budget line (slope $-p_0$)
  - With the bundle (q_0, c_0), she consumes relatively little floor space q and a lot of the consumption good c
- Suburb resident 1 has a flatter budget line (slope $-p_1$) because of the lower intercept $y - tx_1$ to reach the same indifference curve
  - With the bundle (q_1, c_1), she consumes relatively more floor space q and less of the consumption good c

Demand Side: Apartment Size

- In general: How much housing $q(x)$ do residents consume at a given distance x to the city center?

$$\frac{\partial q(x)}{\partial x} = \underbrace{\eta}_{<0} \cdot \underbrace{\frac{\partial p(x)}{\partial x}}_{<0} > 0 \tag{12}$$

  where η is the slope of the demand function.

- **Apartment size per person rises with increasing distance to the city center.** (Result #2)
- $\Longrightarrow$ Suburban apartments have more square meters of floor space than apartments close to the city center.

Demand Side: Total Rent

- Total rent of an apartment is the product of the rental price per square meter times apartment size: $p \cdot q$
- How does total rent depend on distance x to the city center?

$$\frac{\partial [p(x) \cdot q(x)]}{\partial x} = \underbrace{\frac{\partial p(x)}{\partial x}}_{<0} \cdot q(x) + p(x) \cdot \underbrace{\frac{\partial q(x)}{\partial x}}_{>0} \lesseqgtr 0 \tag{13}$$

- $\frac{\partial p(x)}{\partial x} < 0$: On the outskirts, rental prices per square meter are lower...
- $\frac{\partial q(x)}{\partial x} > 0$: but apartments are also larger!
- $\Longrightarrow$ The effect of distance on total rent is ambiguous. A (typically large) apartment on the outskirts can be less or more expensive than a (typically small) apartment in the city center.

Supply Side: Construction Firms

- Construction firms operate under perfect competition.
- They need two input factors to build houses, namely land L and capital K (in effect building materials): $H(L, K)$
- They build houses with a concave production function with constant returns to scale. Concavity implies that one more unit of capital gets them fewer additional floors, the higher the building already is. This makes skyscrapers (and large projects in general) disproportionally expensive to build:

$$\frac{\partial^2 H(L,K)}{\partial K^2} < 0 \tag{14}$$

- Costs: Construction firms rent capital at an exogenously given interest rate i and lease land at the location-specific leasing rate $r(x)$.
- **Profits** are given as revenue minus costs:

$$\Pi = p(x) \cdot H(K, L) - i \cdot K - r(x) \cdot L \tag{15}$$

Supply Side: Building Height

- Let us define $S = \frac{K}{L}$ as the capital-land-ratio. It can be viewed as an index for building height: Increasing capital, while holding land fixed, makes the building higher.
- Then it holds:

$$H(K, L) = L \cdot H\left(\frac{K}{L}, 1\right) = L \cdot H(S, 1) = L \cdot h(S), \tag{16}$$

  where $h(S)$ captures the built-up per unit of land.

- We can write the profit function as

$$\begin{aligned} \Pi &= p(x) \cdot H(K, L) - i \cdot K - r(x) \cdot L \\ &= p(x) \cdot L \cdot h(S) - i \cdot L \cdot S - r(x) \cdot L \qquad (17) \\ &= L \cdot (p(x) \cdot h(S) - i \cdot S - r(x)) \qquad (18) \end{aligned}$$

Supply Side: Profit Maximization

- To find the profit-maximizing building height S depending on x, we take the derivative of the profit function (18) with respect to S and set it equal to zero:

$$\frac{\partial \Pi}{\partial S} = 0 \Longrightarrow p(x) \cdot \frac{\partial h(S)}{\partial S} - i = 0 \tag{19}$$

- Perfect competition implies zero profit:

$$p(x) \cdot h(S) - i \cdot S - r(x) = 0 \tag{20}$$

- For the model solution as a function of x, we compute the total differentials of (19) and (20):

$$\frac{\partial p(x)}{\partial x} \cdot \frac{\partial h(S)}{\partial S} + p(x) \cdot \frac{\partial^2 h(S)}{\partial S^2} \cdot \frac{\partial S}{\partial x} = 0 \tag{21}$$

$$\frac{\partial p(x)}{\partial x} \cdot h(S) + p(x) \cdot \frac{\partial h(S)}{\partial S} \cdot \frac{\partial S}{\partial x} - i \cdot \frac{\partial S}{\partial x} - \frac{\partial r}{\partial x} = 0 \tag{22}$$

Supply Side: Leasing Rate as Negative Function of x

- Plugging $p(x) \cdot \frac{\partial h(S)}{\partial S} = i$ from (19) into (22):

$$\frac{\partial r}{\partial x} = h(S) \cdot \underbrace{\frac{\partial p}{\partial x}}_{<0} < 0 \tag{23}$$

The leasing rate is a decreasing function of distance to the city center. (Result # 3) Land on the outskirts is cheaper to lease for the constrution firms than in the city center.

- $\Longrightarrow$ Construction firms can build at lower costs on land on the outskirts.
- This is again an equilibrium outcome:
  - Same profits (of zero) everywhere, even in the inner city where construction firms can charge a high rent p. $\Longrightarrow$ As the compensating element, the leasing rate r must necessarily be higher in the inner city and lower on the outskirts.

Supply Side: Building Height as a Negative Function of x

- We solve (21) for $\frac{\partial S}{\partial x}$ to see how building height S depends on x:

$$\frac{\partial S}{\partial x} = -\frac{\partial h(S)}{\partial S} \cdot \underbrace{\left(p \cdot \frac{\partial^2 h(S)}{\partial S^2}\right)^{-1}}_{<0} \cdot \underbrace{\frac{\partial p}{\partial x}}_{<0} < 0 \tag{24}$$

The capital-land ratio, as an index for building height, decreases with distance to the city center. (Result # 4)

- In effect, construction firms vary the mix of capital and land to minimize costs:
  - In the inner city, land is relatively more expensive as $r(x)$ is high: $\Longrightarrow$ Less land, more capital $\Longrightarrow$ Higher building heights
  - On the outskirts, land is relatively cheaper as $r(x)$ is low: $\Longrightarrow$ More land, less capital $\Longrightarrow$ Lower building heights

Population Density

- Total built-up H is the product of population N and floor space q per person: $H(x) = N(x) \cdot q(x)$
- Population density $D(x)$ is population per unit of land

$$D(x) = \frac{N(x)}{L} = \frac{H(x)}{L \cdot q(x)} = \frac{h(S)}{q(x)} \tag{25}$$

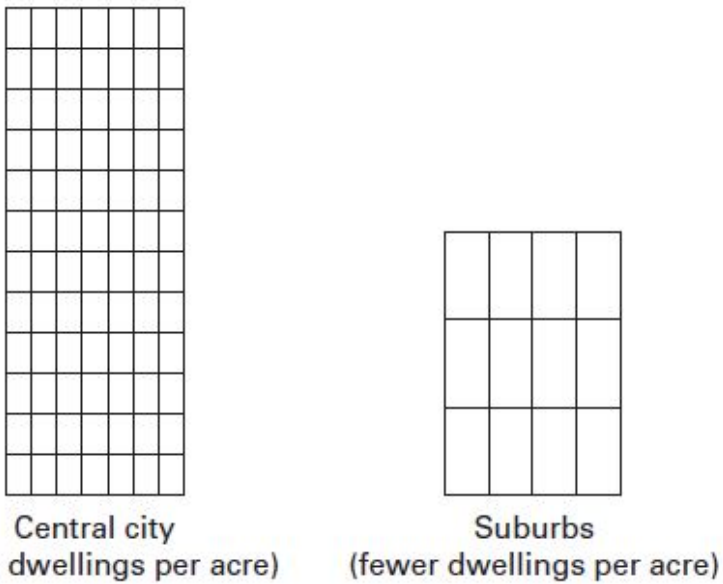
- How does density behave with respect to distance from the city center?

$$\frac{\partial D(x)}{\partial x} = \frac{\partial h(S)}{\partial S} \cdot \underbrace{\left(\frac{\partial S(x)}{\partial x}\right)}_{<0} \cdot \frac{1}{q(x)} - \frac{h(S)}{q(x)} \cdot \underbrace{\frac{\partial q}{\partial x}}_{>0} < 0 \tag{26}$$

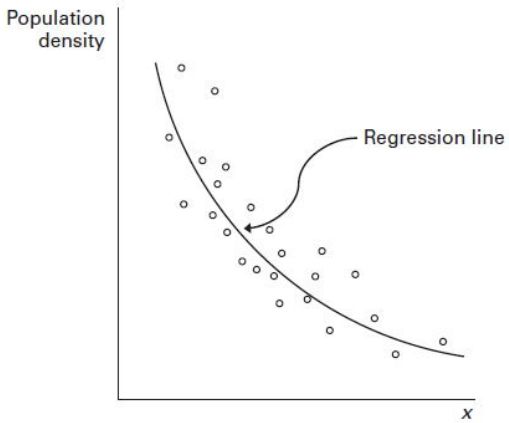
Population density decreases with distance to the city center. (Result # 5)
Two reasons: 1. Building height S decreases, 2. Demanded floor space q per person increases.

Lower Population Density on the Outskirts

- We have two effects which jointly lead to a lower population density on the outskirts
  - 1. Building height is lower on the outskirts
  - 2. Apartments are larger on the outskirts



Population Density



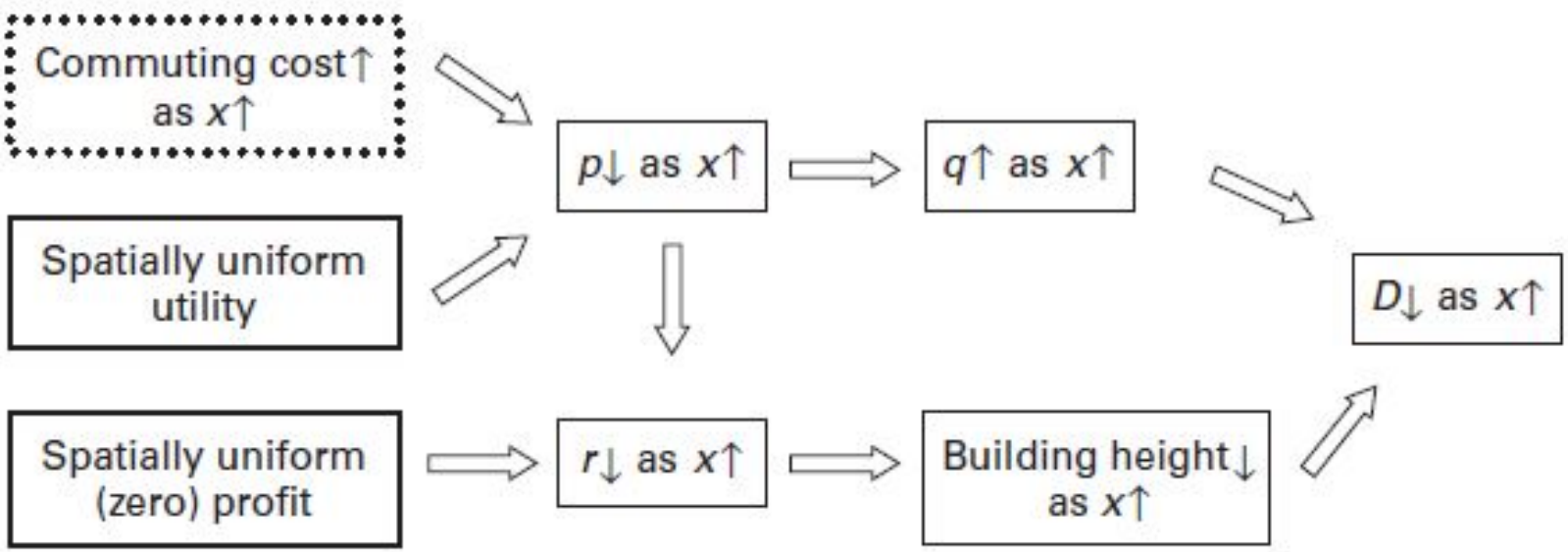
- Population density is often assumed to be an exponential function depending on x. Taking logs yields a linear relation between log population density and x:

$$D = \alpha \cdot exp(-\beta \cdot x) \quad (27)$$

$$log(D) = log(\alpha) - \beta \cdot x \quad (28)$$

- Linear regression between log population density and distance to the city center: We will look at empirical studies in Unit 4

Model Summary



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